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Representation of Supercharges over the Scattering Data for Super-Sine-Gordon System

Shibani Sen[1] and A. Roy Chowdhury[1]

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We determine the form of the supercharges Q_1 and Q_2 in terms of the scattering data for the super-sine-Gordon equation. The technique is to ascertain the effect of supertranslation on the scattering data used in the inverse scattering transform. The same result was obtained by Kulish and Tsypleyv from the Riccati equation by expanding in Λ.

1. INTRODUCTION

With the advent of the inverse scattering transform (IST) for the exact solution of integrable nonlinear equations in two dimensions, it has become possible to analyze various properties of nonlinear systems (Bullough and Caudrey, 1983). One of the most important applications was the analysis of Alonso (1984), who showed how the basic invariance properties of a nonlinear system can be mapped onto the domain of the scattering data. In this analysis he also obtained an interesting characterization of the continuous spectrum, which is really difficult to analyze by any other means Zakharov *et al.*, 1984). Of late the IST also has been formulated for nonlinear equations involving both bosonic and fermionic equations or their coupled supersymmetric systems (D'Hoker, 1983). A very important example is that of the super-sine-Gordon equation (D'Auria, 1980; Chowdhury, 1983). Here we show that it is possible to extend the treatment of Alonso (1984) to the supersymmetric case for obtaining a representation of the supercharges Q_1 and Q_2 in terms of the scattering data. The same result was obtained by Kulish and Tsypleyv (1981) by use of the Riccati equation. Our treatment explicitly demonstrates the use of supersymmetric transformations in the mapping of the scattering data to physical quantities.

[1]High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032 India.

2. FORMULATION

The super-sine-Gordon system is written as

$$(r^{\mu}\partial_{\mu}+m\cos\beta\phi)\psi=0$$
$$\Box\phi=\tfrac{1}{2}[(m^2/\beta)\sin 2\beta\phi-im\beta\bar{\psi}\psi\cos\beta\phi] \tag{1}$$

where ϕ is the bosonic field and ψ_1 and ψ_2 are the fermionic counterparts. These are really the components of the scalar superfluid $\Phi(\theta, x)$, written as

$$\Phi(\theta, x, t)=\phi(x)+i\bar{\theta}\psi(x)+\tfrac{1}{2}i\bar{\theta}\theta F(x) \tag{2}$$

where $\theta=(\theta_1, \theta_2)^t$ denotes the fermionic coordinates and (x, t) the ordinary two-dimensional space-time. In the supervariables, equations (1) can be compactly written as

$$D_1 D_2\Phi=\sin\Phi \tag{3}$$

D_1, D_2 stand for the covariant derivatives and are written as

$$D_1=\frac{\partial}{\partial\theta_2}-i\theta_2(\partial_x+\partial_t);\qquad D_2=-\frac{\partial}{\partial\theta_1}-i\theta_1(\partial_x-\partial_t) \tag{4}$$

Equation (3) is invariant under various supertransformations, of which supertranslations form a small subset, defined via the rules

$$x'^{\mu}=x^{\mu}-i\bar{\varepsilon}\gamma^{\mu}\theta$$
$$\bar{\varepsilon}=(\varepsilon^1, \varepsilon^2) \tag{5}$$
$$\theta'=\theta+\varepsilon$$

where ε^1, ε^2 are anticommuting parameters. The inverse scattering equations associated with SSG are

$$D_1\chi=(M+\tfrac{1}{2}\tilde{I}\Lambda^2\theta_2)\chi$$
$$D_2\chi=(i\Lambda^{-1}N)\chi \tag{6}$$

where

$$M=\begin{pmatrix} 0 & D_1\Phi & 0\\ -D_1\Phi & 0 & -\Lambda\\ 0 & \Lambda & 0\end{pmatrix} \tag{7}$$

$$N=\begin{pmatrix} 0 & 0 & \sin\Phi\\ 0 & 0 & -\cos\Phi\\ \sin\Phi & -\cos\Phi & 0\end{pmatrix} \tag{8}$$

$\tilde{I}$ stands for a 3×3 unit matrix.

Next it is interesting to note that it is really difficult to have a physically meaningful interpretation of the scattering process in case of anticommuting coordinates (θ_1, θ_2). For this it is always advisable to separate the ordinary and supercomponents of equations (6)-(8). For this we set

$$\chi(\theta, x, t) = (\tilde{I} + i\theta_1 A + i\theta_2 B + i\theta_1\theta_2 C)\psi(x, t) \tag{9}$$

where ψ satisfies a Lax pair written in ordinary coordinates. In equation (9) the matrices A, B, and C are

$$A = \Lambda^{-1}\begin{pmatrix} 0 & 0 & \sin\phi \\ 0 & 0 & \cos\phi \\ \sin\phi & \cos\phi & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\psi_1 & 0 \\ \psi_1 & 0 & -i\Lambda \\ 0 & i\Lambda & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & \sin\phi & -i\Lambda^{-1} & \cos\phi\psi_1 \\ 0 & -\cos\phi & +i\Lambda^{-1} & \sin\phi\psi_1 \\ 0 & 0 & \cos\phi & \end{pmatrix} \tag{10}$$

For future reference we quote here the ordinary Lax pair for ψ,

$$i\psi_\xi = \begin{pmatrix} -\Lambda^2/2 & i\partial_\xi\phi & -i\Lambda\psi_1 \\ -i\partial_\xi\phi & \Lambda^2/2 & 0 \\ i\Lambda\psi_1 & 0 & \Lambda^2/2 \end{pmatrix}$$

$$i\psi_\eta = \Lambda^{-2}\begin{pmatrix} \sin^2\phi & \sin\phi\cos\phi & \Lambda\cos\phi\psi_2 \\ \sin\phi\cos\phi & \cos^2\phi & -\Lambda\sin\phi\psi_2 \\ \Lambda\cos\phi\psi_2 & -\Lambda\sin\phi\psi_2 & 1 \end{pmatrix} \tag{11}$$

(ξ, η) are light cone variables.

The inverse scattering theory for the super-sine-Gordon equation is formulated through the definition of the Jost solutions. From equation (11) we observe that for the soliton modes, since $\phi, \psi_1, \psi_2 \to 0$ as $\xi \to \pm\infty$, the Jost solution is defined by the asymptotic condition

$$\psi^{\pm}(\xi, \Lambda) \underset{\xi\to\pm\infty}{\longrightarrow} \exp[-iL_0(\Lambda)\xi]$$

$$L_0(\Lambda) = \tfrac{1}{2}\Lambda^2\,\mathrm{diag}(-1, 1, 1), \qquad \xi = \tfrac{1}{2}(x + t)$$

So the asymptotically the super-Jost function $\chi(\theta, x, t)$ is written as

$$\chi(\theta, x, t) = \begin{pmatrix} e^{i\Lambda^2(x+t)/2} & 0 & 0 \\ 0 & 1 - i\theta_1\theta_2 & i\theta_1/\Lambda + \Lambda\theta_2 \\ 0 & i\theta_1/\Lambda - \Lambda\theta_2 & 1 + i\theta_1\theta_2 \end{pmatrix} e^{-i\Lambda^2(x+t)/4} \tag{12}$$

3. THE SUPERTRANSLATION

The supertransformations (5) written in detail read

$$\begin{aligned} x' &= x - i(\varepsilon^1\theta_2 + \varepsilon^2\theta_1) \\ t' &= t - i(\varepsilon^1\theta_2 - \varepsilon^2\theta_1) \\ \theta_1' &= \theta_1 + \varepsilon^2 \\ \theta_2' &= \theta_2 - \varepsilon^1 \end{aligned} \tag{13}$$

Then the operator X_1 changes according to

$$X'_{1,\mathrm{as}} = X_{1,\mathrm{as}} + 2i\varepsilon^1(\partial_x + \partial_t)\tilde{I} + \tfrac{1}{2}\Lambda^2\varepsilon^1\tilde{I}$$

where we have considered only the asymptotic forms of X_1 and X'_1. Now the new or changed eigenfunction χ' should satisfy

$$\left[X'_1 + \begin{pmatrix} \varepsilon^1\Lambda^2 & 0 & 0 \\ 0 & -\varepsilon^1\Lambda^2 & 0 \\ 0 & 0 & -\varepsilon^1\Lambda^2 \end{pmatrix}\right]\chi'_{\mathrm{as}} = 0 \tag{14}$$

Similarly, X_2 changes to

$$X'_{2,\mathrm{as}} = X_{2,\mathrm{as}} - 2i\varepsilon^2(\partial_x - \partial_t)\tilde{I} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon^2/\Lambda^2 & 0 \\ 0 & 0 & \varepsilon^2/\Lambda^2 \end{pmatrix} \tag{15}$$

and χ'_{as} also satisfies

$$\left[X'_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\varepsilon^2/\Lambda^2 & 0 \\ 0 & 0 & -\varepsilon^2/\Lambda^2 \end{pmatrix}\right]\chi'_{\mathrm{as}} = 0 \tag{16}$$

It is not difficult to see that the most general structure of χ' can be written as

$$\chi' = \begin{pmatrix} 1 - \varepsilon^1\Lambda^2\theta_2 & 0 & 0 \\ 0 & 1 - \tfrac{1}{2}\varepsilon^1\varepsilon^2 & -\tfrac{1}{2}(i\varepsilon^2/\Lambda - \varepsilon^1\Lambda) \\ 0 & -\tfrac{1}{2}(i\varepsilon^2/\Lambda + \varepsilon^1\Lambda) & 1 + \tfrac{1}{2}i\varepsilon^1\varepsilon^2 \end{pmatrix}\chi \tag{17}$$

On the other hand, the supertranslation is generated by D_1, D_2, Q_1, and Q_2, and the corresponding operator is

$$N = \exp(\varepsilon_1 Q_1)\exp(\varepsilon_2 Q_2)\exp(\delta_1 D_1)\exp(\delta_2 D_2) \tag{18}$$

with

$$\delta_1 = -i(\varepsilon^1\theta_2 - \varepsilon^2\theta_1), \qquad \delta_2 = -i(\varepsilon^1\theta_2 + \varepsilon^2\theta_1) \tag{19}$$

So we can express the left-hand side of equation (17) and hence the change in χ, i.e., $\chi'-\chi=\delta\chi$, as

$$\begin{aligned}\delta\beta x = &-i\varepsilon^1\theta_2[(D_1+D_2),\chi]+i\varepsilon^2\theta_1[(D_1-D_2),\chi]\\ &+i\varepsilon^2[Q_1,\chi]-\varepsilon^1[Q_2,\chi]-i\varepsilon^1\varepsilon^2\theta_1\\ &\times[Q_2,[D_1-D_2,\chi]]_+ + i\varepsilon^1\varepsilon^2\theta_2[Q_1,[D_1+D_2,\chi]]\\ &+\varepsilon^1\varepsilon^2[Q_1,[Q_2,\chi]]_+ \end{aligned} \tag{20}$$

where $[\,,]_+$ denotes the anticommutator.

Now taking the limit $x\to+\infty$, we get [in equation (20)]

$$\delta\chi=\begin{pmatrix} -\varepsilon^1\Lambda^2\theta_2 & 0 & 0\\ 0 & -\frac{1}{2}i\varepsilon^1\varepsilon^2 & -(i\varepsilon^2/\Lambda-\varepsilon^1\Lambda)\\ 0 & -\frac{1}{2}(i\varepsilon^2/\lambda+\varepsilon^1\Lambda) & \frac{1}{2}i\varepsilon^1\varepsilon^2 \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{pmatrix} \tag{21}$$

where T_{ij} are the scattering matrices defined through the two sets of Jost functions. Equating the two expressions for $\delta\chi$, we obtain

$$\begin{aligned}[Q_2,\,T_{2i}]&=-\tfrac{1}{2}\Lambda T_{3i}\\ [Q_2,\,T_{3i}]&=\tfrac{1}{2}\Lambda T_{2i}\end{aligned} \tag{22}$$

Also,

$$\begin{aligned}[Q_1,\,T_{3i}]_+&=\frac{i}{\Lambda}T_{2i}\\ [Q_1,\,T_{3i}]_-&=-\frac{i}{2\Lambda}T_{2i}\end{aligned} \tag{23}$$

From equations (22) and (23) we immediately observe that we can write the supercharges Q_1 and Q_2 as

$$Q_2=-\tfrac{1}{4}\Lambda T_{3i}T_{2i}^{-1},\qquad Q_1=(i/4\Lambda)T_{3i}T_{2i}^{-1}$$

But since from physical considerations it follows that Q_1 and Q_2 should be independent of Λ, it follows that appropriate limits in Λ should be taken on the right-hand sides of these expressions.

4. DISCUSSION

In our above analysis we have shown that it is possible to have representations of the supercharges in terms of the scattering data, by considering the effect of a supertransformation over the Jost functions at positive and negative infinity. It is interesting to observe that the machinery of IST is quite versatile in analyzing the invariance properties of nonlinear systems.

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